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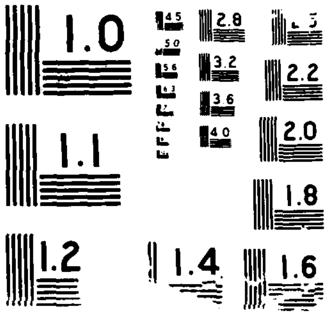
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CMS Technical Summary Report #88-13

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November 1987

(Received November 6, 1987)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Air Force Office of Scientific Research
Bolling Air Force Base
Washington, DC 20332

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

HYPERBOLIC CONSERVATION LAWS IN VISCOELASTICITY

J. A. Nohel[1], R. C. Rogers[2], and A. E. Tzavaras[3]

1. Introduction.

The equations of unsteady motion for nonlinear elastic materials are quasilinear systems of hyperbolic equations in which characteristic speeds are not constant. Thus weak initial waves are amplified and smooth solutions generally develop singularities in finite time. Particularly interesting situations arise when this destabilizing mechanism coexists and competes with dissipation; a simple example is provided by the quasilinear wave equation with linear first-order damping. A more subtle dissipation occurs in certain viscoelastic materials such as polymers, suspensions and emulsions which have **memory**; i.e. the stress at each material point depends not only on the present value of the deformation gradient (and/or velocity gradient), but on the entire temporal history of motion. These materials exhibit behavior intermediate between that of an elastic solid and a viscous fluid. Generally, the memory fades with time: disturbances which occurred in the distant past have less influence on the present stress than those which occurred in the recent past. ~~Our~~ purpose is to discuss qualitative properties of the equations which model unsteady motions of such materials. In order to present some of the central ideas while avoiding serious technical difficulties, ~~we~~ shall restrict the discussion to a particular model problem for the motion of a one-dimensional viscoleastic material with fading memory. Many of the qualitative features in this relatively simple situation are present in more general integral

[1]Partially supported by the United States Army under Contract No. DAAG29-80-C-0041 and Grant No. DAAL03-87-K-0036, by the United States Air Force under Grant No. AFOSR-87-0191, and by the National Science Foundation under Grant No. DMS-8620303.
[2]Partially supported by the United States Army under Contract No. DAAG29-80-C-0041, and by the National Science Foundation under Grant No. DMS-210950.
[3]Partially supported by the National Science Foundation under Grant No. DMS-8501397, and by the United States Army under Contract No. DAAG29-80-C-0041.

and differential models in one and in several space dimensions; these are discussed in a forthcoming monograph [37].

Motivated by the study of the growth and decay of acceleration waves by Coleman and Gurtin ([5], [6]) and by experiments ([34], [43]), it is natural to require that reasonable mathematical models for unsteady motion of viscoelastic materials should possess qualitative properties which include:

(a) Under physically reasonable assumptions, the memory should induce a weak dissipative mechanism. As a consequence, for sufficiently smooth and small data and history the equations of motion should have globally defined smooth solutions which decay as $t \to \infty$.

(b) By contrast, if the smooth data and history are chosen sufficiently large, the classical solution solutions should break down in finite time and exhibit a shock structure.

(c) For rough, bounded data, the equations of motion should possess globally defined weak solutions.

This paper is organized as follows. In Section 2, we state the model problem and related, well-understood problems, and we briefly review known results regarding properties (a) and (b) for classical solutions. We then state a new result [32] on the existence of a weak solution (in the class of bounded measurable functions) which verifies property (c) in an important special case. In Section 3, we sketch the proof of the new result using the methods of vanishing viscosity and compensated compactness; further details can be found in [32].

2. Model Problem and Summary of Results.

We study the model Cauchy problem for the system of integrodifferential equations

$$\begin{cases} w_t = v_x, \ v_t = \sigma_x, & x \in \mathbf{R}, \ t > 0, \\ w(x,0) = w_0(x), \ v(x,0) = v_0(x), & x \in \mathbf{R}, \end{cases} \tag{VE}$$

where the function $\sigma(x,t)$ is determined by the history of $w(x,\cdot)$ through the constitutive assumption

$$\sigma(x,t) = \varphi(w(x,t)) + \int_0^t k(t-\tau)\psi(w(x,\tau))d\tau, \tag{CA}$$

which generalizes Boltzmann's law for linear viscoelasticity [3]. The given functions $\varphi(w)$, $\psi(w)$ and $k(t)$ are assumed to be smooth and, in addition,

$$\varphi'(w) > 0, \quad w \in \mathbf{R},$$

so that the structure of (VE) is hyperbolic. The system (VE) is a model for one-dimensional motion of a viscoelastic material with fading memory. The functions $v(x,t), w(x,t)$ and $\sigma(x,t)$ stand for the velocity, deformation gradient and stress, respectively, while the constitutive assumption (CA) states that the stress is a particular functional of the history of the deformation gradient; in (CA) the history of $w(x,\cdot)$ is assumed to be zero for $t < 0$ and the body force is taken to be zero. Under appropriate assumptions on the kernel k, the memory induces a weak dissipative mechanism which competes with the hyperbolic character of (VE). In the sequel, we will limit our discussion to the Cauchy problem. We remark that when x is restricted to a finite interval, boundary conditions which are compatible with the data w_0, v_0 must be adjoined to (VE).

It is useful to consider several special cases. If $k \equiv 0$, (VE) reduces to the system of conservation laws

$$\begin{cases} w_t = v_x, \ v_t = \varphi(w)_x \\ w(x,0) = w_0(x), \ v(x,0) = v_0(x), \end{cases} \tag{E}$$

which describes the motion of one-dimensional elastic materials. The solutions of (E) generally develop singularities in finite time no matter how smooth and small the data w_0, v_0 are taken ([19],[24],[18]). At the other extreme, if one formally sets $k = -\delta'$, where δ is the Dirac mass at the origin, (VE) reduces to the system

$$w_t = v_x, \ v_t = \psi(w)_{xt} + \varphi(w)_x,$$

which is parabolic and which possesses globally defined smooth solutions even for the large data w_0, v_0, whenever ψ is smooth and $\psi'(\cdot) > 0$ ([1],[21]). The theory of (VE) lies between these two extremes. In the special case $\psi \equiv \phi$, (VE) reduces to the system

$$\begin{cases} w_t = v_x, \\ v_t = \varphi(w)_x + \int_0^t k(t-\tau)\varphi(w(x,\tau))_x d\tau, & x \in \mathbf{R}, \ t > 0, \\ w(x,0) = w_0(x), \ v(x,0) = v_0(x), & x \in \mathbf{R}, \end{cases} \tag{2.1}$$

which, as we shall see, exhibits a close resemblance to the theory of the system

$$\begin{cases} w_t = v_x \\ v_t = \varphi(w)_x - v. \end{cases} \tag{FE}$$

(FE) describes the motion of a one-dimensional frictionally damped elastic material and enjoys properties (a), (b) and (c) which are established in [29,38,13] respectively. The similarity between (2.1) and (FE) is revealed by the following idea of MacCamy [22]. Let $r(t)$ be the resolvent kernel associated with k; i.e., r is the solution of the linear Volterra equation

$$r(t) + \int_0^t k(t-\tau)r(\tau)d\tau = k(t), \quad t \geq 0. \tag{r}$$

Convolving $(2.1)_2$ with $r(t)$, a simple calculation yields

$$\begin{aligned} \int_0^t k(t-\tau)\varphi(w(x,\tau))_x d\tau &= \int_0^t r(t-\tau)v_t(x,\tau)d\tau \\ &= r(0)v(x,t) - r(t)v_0(x) + \int_0^t r'(t-\tau)v(x,\tau)d\tau. \end{aligned} \tag{2.2}$$

Thus, for classical solutions (2.1) is equivalent to

$$\begin{cases} w_t = v_x, \ v_t = \varphi(w)_x + \mathcal{F}[v], & x \in \mathbf{R}, \ t > 0, \\ w(x,0) = w_0(x), \ v(x,0) = v_0(x), & x \in \mathbf{R}, \end{cases} \tag{2.3}$$

where

$$\mathcal{F}[v](x,t) := r(0)v(x,t) - r(t)v_0(x) + \int_0^t r'(t-\tau)v(x,\tau)d\tau. \tag{2.4}$$

Under physically reasonable assumptions (e.g., $k = a'$, where $a : [0,\infty) \to \mathbf{R}$ is smooth, positive, decreasing and convex, $r(0) = a'(0) < 0$) the term $r(0)v(x,t)$ has a damping effect. Its effect is dominant close to equilibrium ($w \equiv 0, v \equiv 0, \phi(0) = 0$), thereby inducing property (a) ([22],[11],[39]). However, for suitably large data, the hyperbolic part of (2.1) dominates irrespective of the sign of the memory, and property (b) holds if ϕ is convex or concave ([8],[31],[35]).

To discuss the results for classical solutions of (VE) when $\psi \neq \phi$, we assume that the smooth constitutive functions ϕ, ψ defined on $\mathbf{R}$ satisfy

$$\phi(0) = \psi(0) = 0, \ \phi'(\cdot) > 0, \ \psi'(0) > 0.$$

To simplify the exposition, we also assume that the kernel $k = a'$, where $a : [0,\infty) \to \mathbf{R}$ is a smooth, positive, nonincreasing, and convex function on $[0,\infty)$; such kernels are relevant for applications in viscoelasticity, (more generally, one can assume that a is strongly positive on $[0,\infty)$, [33],[11],[12],[16],[39]). The following results regarding the behavior of classical solutions of (VE) have been established under additional technical assumptions which are omitted:

(a) If $\phi'(0) - a(0)\psi'(0) > 0$ (i.e., the equilibrium stress modulus is positive) and if the initial data w_0, v_0 are sufficiently smooth and small, it is shown in [12], [16] using delicate energy estimates and properties of Volterra kernels that (VE) has a globally defined smooth solution which decays as $t \to +\infty$. Similar global results for more general models are discussed in [15]. [37; Ch. IV].

(b) By contrast, for sufficiently large, smooth data w_0, v_0 and smooth kernels k irrespective of sign properties, the solutions of (VE) develop singularities in finite time; the first derivatives of w and v become infinite while w, v remain bounded. This suggests the formation of a shock front. Such results are established for ϕ convex or concave by the method of characteristics in [8] and [31]. Similar results for other integral models with smooth memory kernels and for several differential models are given in [37], Chapter II. Numerical evidence of finite-time breakdown of smooth solutions for large data and for the onset of a shock structure for (VE) are provided in [26].

A parallel theory regarding properties (a) and (b) has been developed in [30], [25], [7], [23] for the simpler model Cauchy problem for the single conservation law with memory

$$w_t + \sigma_x = 0, \; w(x,0) = w_0(x), \qquad (CLM)$$

where σ is given by (CA). When $k \equiv 0$, (CLM) reduces to the Burgers equation

$$w_t + \phi(w)_x = 0, \qquad (B)$$

while the analogue of (FE) is

$$w_t + \phi(w)_x + w = 0. \qquad (FB)$$

Our discussion of (VE) is limited to kernels which are smooth on $[0,\infty)$. However, we remark that a result confirming property (a) for (VE) with a singular kernel ($k = a'$,

a positive, decreasing, convex on $[0,\infty)$, and a' having an integrable singularity at zero) has been established by Hrusa and Renardy [17] (also see these proceedings, paper by Renardy), if x is confined to a finite interval; it is not known whether property (a) holds for the corresponding Cauchy problem. Moreover, it is not known whether property (b) holds for (VE) with such singular kernels.

The results (b) discussed above provide a motivation for studying weak solutions of (VE). So far, only special results concerning weak solutions of (VE) have appeared in the literature. Greenberg [14] (also see [37], Section 2.6) established the existence of travelling wave solutions (steady compression shocks) for a history value problem associated with (VE). Regarding unsteady weak solutions, we present a new existence result for the Cauchy problem (2.1) when the data $w_0, v_0 \in L^\infty(\mathbf{R})$, and not necessarily small.

We assume: the constitutive function φ satisfies

$$\begin{cases} \varphi : \mathbf{R} \to \mathbf{R} \text{ is a twice continuously differentiable} \\ \text{function such that } \varphi'(w) > 0,\ w \in \mathbf{R}; \\ \varphi \text{ has a single inflection point at } w = w_i \text{ and is} \\ \text{convex on } (w_i, \infty) \text{ and concave on } (-\infty, w_i). \end{cases} \tag{2.5}$$

The kernel k satisfies

$$k : [0,\infty) \to \mathbf{R}, \quad k \in C^1[0,\infty), \tag{2.6}$$

and the data $w_0(x), v_0(x)$ satisfy

$$w_0(x),\ v_0(x) \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}). \tag{2.7}$$

The following result establishes property (c) for the system (2.1).

Theorem 2.1. *Let the hypotheses (2.5)-(2.7) be satisfied. Given $T > 0$, there exists a weak solution $\{w(x,t), v(x,t)\}$ of (2.1) on $\mathbf{R} \times [0,T]$, such that*

$$(w,v) \in L^\infty([0,T]; L^2(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [0,T]).$$

The proof is sketched in Section 3 using the method of compensated compactness of Murat [27] and Tartar [40,41,42]; complete details of the proof will appear in [32]. This approach has been employed with success by Tartar [41] to obtain L^∞ solutions for the general, scalar Burgers equation (B), by DiPerna [13] and Rascle [36] to construct L^∞

solutions to the hyperbolic system (E), and by Dafermos [10] to obtain L^∞ solutions for (CLM); unfortunately, the elegant approach of [10] does not seem to apply to (VE) or (2.1). We also remark that Boldrini [2] used the method of compensated compactness to show that as the memory weakens (i.e., in (2.1) $k = k(\delta, t) = 0(\delta)$), uniformly bounded solutions $\{w^\delta, v^\delta\}$ of (2.1) converge to a weak solution of (E) as $\delta \downarrow 0$.

Serious difficulties remain to be overcome in order to prove a result similar to Theorem 2.1 for (VE) when $\psi \neq \phi$. The central issue is whether the memory term $\int_0^t k(t-\tau)\psi(w)(x,\tau))_x d\tau$ can be shown to be of lower order; such is shown to be the case if $\psi \equiv \phi$, even for weak solutions.

3. Sketch of the Proof of Theorem 2.1.

The weak solutions of (2.1), in the space of bounded measurable functions, will be constructed as limits of solutions of the regularized (parabolic) system

$$\begin{cases} w_t = v_x + \varepsilon w_{xx} \\ v_t = \varphi(w)_x + \int_0^t k(t-\tau)\varphi(w(x,\tau))_x d\tau \\ \qquad\qquad + \varepsilon(v_{xx} + \int_0^t k(t-\tau) v_{xx}(x,\tau)d\tau), \quad (x,t) \in \mathbf{R} \times (0,T] \\ \\ w(x,0) = w_0(x), \; v(x,0) = v_0(x), \qquad\qquad x \in \mathbf{R}, \end{cases} \tag{3.1}$$

as the real parameter $\varepsilon \downarrow 0$. This unconventional regularization preserves one of the main features of (2.1); namely, that the integration with respect to t offsets differentiation with respect to x and the memory terms are of lower order. Its relevance is revealed by the following calculation. Convolving $(3.1)_2$ with $r(t)$, and using equations (r) and (2.4), we find

$$\begin{aligned} \int_0^t k(t-\tau)(\varphi(w(x,\tau))_x + \varepsilon v_{xx}(x,\tau))d\tau &= \int_0^t r(t-\tau)v_t(x,\tau)d\tau \\ &= r(0)v(x,t) - r(t)v_0(x) + \int_0^t r'(t-\tau)v(x,\tau)d\tau = \mathcal{F}[v](x,t). \end{aligned} \tag{3.2}$$

Thus, the initial value problem (3.1) can be written in the form

$$\begin{cases} w_t = v_x + \varepsilon w_{xx}, \; v_t = \varphi(w)_x + \mathcal{F}[v] + \varepsilon v_{xx}, \quad (x,t) \in \mathbf{R} \times (0,T] \\ \\ w(x,0) = w_0(x), \; v(x,0) = v_0(x), \quad x \in \mathbf{R}. \end{cases} \tag{3.3}$$

It is evident that systems (3.1) and (3.3) are equivalent for smooth data and classical solutions. For the data in $L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$, the above ideas extend to weak solutions; i.e., it is shown in [32] that (3.1) and (3.3) (also (2.1) and (2.3)) are equivalent for L^∞ solutions, using C^1-test functions which have compact support on $\mathbf{R} \times [0,T]$ for any $T > 0$ and which vanish at $t = T$. Moreover, (2.2) and (3.2) hold in the weak sense.

Our next objective is to obtain a-priori estimates, independent of ε, for solutions $\{w^\varepsilon, v^\varepsilon\}$ of (3.1) corresponding to initial data $w_0, v_0 \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$; we will employ the equivalent system (3.3). The following result is needed to complete the proof of Theorem 2.1.

Theorem 3.1. *Under the hypotheses (2.5)-(2.7), for each $\varepsilon > 0, T > 0$, the initial value problem (3.3) (respectively (3.1)) has a unique solution $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}$ defined on $R \times [0,T]$ such that $w^\varepsilon, v^\varepsilon \in C([0,T]; L^2(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [0,T])$, $w^\varepsilon_x, v^\varepsilon_x, w^\varepsilon_{xx}, v^\varepsilon_{xx}, w^\varepsilon_t, v^\varepsilon_t \in C((0,T]; L^2(\mathbf{R}))$ and, also, $w^\varepsilon_x, v^\varepsilon_x \in L^2(\mathbf{R} \times [0,T])$. In addition, as $\varepsilon \downarrow 0$, the families $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}_{\varepsilon>0}$ and $\{\varepsilon^{1/2} w^\varepsilon_x(x,t), \varepsilon^{1/2} v^\varepsilon_x(x,t)\}_{\varepsilon>0}$ lie in bounded sets of $L^\infty([0,T]; L^2(\mathbf{R}) \cap L^\infty(\mathbf{R}))$ and $L^2(\mathbf{R} \times [0,T])$, respectively.*

The existence, uniqueness and regularity properties of the solution $(w^\varepsilon, v^\varepsilon)$ of (3.3) are established in [32] by a fairly standard fixed-point argument on an appropriately chosen Banach space using energy methods and the L^2-theory of the heat equation. We will sketch the proof of the a-priori estimates; we emphasize that due to the nonlocal nature of the memory-term in (3.3), the L^∞-estimates cannot be obtained by finding an invariant region [4] (compare with the proof of the existence of weak solutions for (E) in [13]). To avoid a confusing notation, we now drop the superscript ε.

The a-priori estimates are deduced by using a class of exponentially growing convex entropies for (E) constructed by Dafermos [9]. Let $\{w(x,t), v(x,t)\}$ be the solution of (3.3) on $\mathbf{R} \times [0,T]$ satisfying the regularity properties of Theorem 3.1; this assumption justifies the calculations which follow. In the sequel, C will stand for a generic constant depending on the $L^\infty(\mathbf{R})$ and $L^2(\mathbf{R})$-norms of the initial data, on the $C^1[0,T]$-norm of $r(t)$, on properties of the function $\varphi(w)$, on T, but not on ε. Whenever the constant depends on ε it will be denoted by C_ε. The analysis of (3.3) will be based on the concept of entropy-entropy flux pairs for the elastic problem (E) (cf. Lax [20]). A smooth, convex

function $\eta(w,v)$ defined on $\mathbf{R}\times\mathbf{R}$ is an entropy for (E), with corresponding entropy flux $q(w,v)$, if

$$\partial_t \eta(w(x,t),v(x,t)) + \partial_x q(w(x,t),v(x,t)) = 0 \tag{3.4}$$

for any smooth solution $\{w(x,t),v(x,t)\}$ of (E). Such pairs are generated as solutions of the system of equations

$$\begin{cases} q_w = -\varphi'(w)\eta_v \\ q_v = -\eta_w, \end{cases} \tag{3.5}$$

provided $\eta(w,v)$ is convex. Eliminating $q(w,v)$ in (3.5), we find that $\eta(w,v)$ must be a convex solution of the linear wave equation

$$\eta_{ww} = \varphi'(w)\eta_{vv}; \tag{3.6}$$

$q(w,v)$ is then determined by (3.5). A classical example of an entropy-entropy flux pair is

$$\eta(w,v) = \frac{1}{2}v^2 + \int_0^w \varphi(\xi)d\xi, \quad q(w,v) = -v\varphi(w). \tag{3.7}$$

For each entropy-entropy flux pair $\{\eta(w,v),q(w,v)\}$ for (E), we denote by $\{\bar{\eta}(w,v), \bar{q}(w,v)\}$ the parts of the pair $\{\eta,q\}$ which vanish to quadratic order in (w,v). Then $\{\bar{\eta},\bar{q}\}$ also form an entropy-entropy flux pair for (E), and a simple computation yields the identity

$$\begin{aligned} \partial_t\bar{\eta}(w,v) + \partial_x\bar{q}(w,v) &= \bar{\eta}_v(w,v)\mathcal{F}[v] + \varepsilon\partial_x^2\bar{\eta}(w,v) \\ &- \varepsilon[\eta_{ww}(w,v)w_x^2 + 2\eta_{wv}(w,v)w_xv_x + \eta_{vv}(w,v)v_x^2]. \end{aligned} \tag{3.8}$$

Integrating (3.8) over $\mathbf{R}\times(0,t)$, $0 < t \leq T$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty}\bar{\eta}(w(x,t),v(x,t))dx \\ &\quad + \varepsilon\int_0^t\int_{-\infty}^{\infty}(\eta_{ww}(w,v)w_x^2 + 2\eta_{wv}(w,v)w_xv_x + \eta_{vv}(w,v)v_x^2)dxd\tau \\ &\quad = \int_{-\infty}^{\infty}\bar{\eta}(w_0(x),v_0(x))dx + \int_0^t\int_{-\infty}^{\infty}\bar{\eta}_v(w,v)\mathcal{F}[v]dxd\tau, \quad 0\leq t\leq T. \end{aligned} \tag{3.9}$$

For the first estimate, we employ the pair (3.7) and use (3.9) to deduce

$$\begin{aligned} &\int_{-\infty}^{\infty}[\frac{1}{2}v^2(x,t) + \Phi(w(x,t))]dx + \varepsilon\int_0^t\int_{-\infty}^{\infty}(\varphi'(w)w_x^2 + v_x^2)dxd\tau \\ &\quad = \int_{-\infty}^{\infty}[\frac{1}{2}v_0^2(x,t) + \Phi(w_0(x,t))]dx \\ &\qquad + \int_0^t\int_{-\infty}^{\infty}v\mathcal{F}[v]dxd\tau, \ 0\leq t\leq T, \end{aligned} \tag{3.10}$$

w' ere

$$\Phi(w) := \int_0^w (\varphi(\xi) - \varphi(0))d\xi. \tag{3.11}$$

Assumption (2.5) implies that

$$\varphi'(w) \geq \varphi'(w_i) > 0, \text{ and } \Phi(w) \geq \frac{\varphi'(w_i)}{2} w^2 \geq 0, \text{ for every } w \in \mathbf{R}. \tag{3.12}$$

Using (2.4) and the Cauchy-Schwarz inequality, a simple calculation yields

$$\int_0^t \int_{-\infty}^{\infty} v\mathcal{F}[v]dxd\tau \leq C(1 + \int_0^t \int_{-\infty}^{\infty} v^2(x,\tau)dxd\tau), \quad 0 \leq t \leq T. \tag{3.13}$$

Combining (3.10) with (3.12) and (3.13), gives

$$\int_{-\infty}^{\infty} v^2(x,t)dx \leq C + C\int_0^t \int_{-\infty}^{\infty} v^2 dxd\tau, \tag{3.14}$$

whence by Gronwall's inequality, (3.10) and (3.12),

$$\begin{aligned} &\int_{-\infty}^{\infty} [v^2(x,t) + w^2(x,t)]dx \\ &\qquad + \varepsilon \int_0^t \int_{-\infty}^{\infty} (w_x^2 + v_x^2)dxd\tau \leq C, \quad 0 \leq t \leq T, \end{aligned} \tag{3.15}$$

where C is independent of ε. Thus $\{w(x,t), v(x,t)\}$ lies in a bounded set of $L^\infty([0,T]; L^2(\mathbf{R}))$ and $\{\varepsilon^{1/2}w_x(x,t), \varepsilon^{1/2}v_x(x,t)\}$ lies in a bounded set of $L^2(\mathbf{R} \times [0,T])$, independent of ε.

The proof of the L^∞-estimates independent of ε is more subtle. For this purpose, v extend the development of Dafermos [9]. The following facts are proved in [9, Section 2]: For $\varphi(w)$ as in (2.5), the wave equation (3.6) admits a class of solutions $\{\eta^{(k)}(w,v)\}_{k>0}$ on $\mathbf{R}\times\mathbf{R}$ which are strictly convex and grow exponentially at infinity. These solutions have the form

$$\eta^{(k)}(w,v) = Y^{(k)}(w) cosh\ kv, \quad 0 < k < \infty, \tag{3.16}$$

where $Y^{(k)}(w)$ is the solution of the initial value problem

$$\begin{cases} Y^{(k)''}(w) = k^2\varphi'(w)Y^{(k)}(w), \\ Y^{(k)}(w_i) = 1, \ Y^{(k)'}(w_i) = 0, \quad 0 < k < \infty. \end{cases} \tag{3.17}$$

The functions $Y^{(k)}(w)$ satisfy the estimates

$$Y^{(k)}(w) \geq cosh[k\sqrt{\varphi'(w_*)}(w - w_*)], \quad -\infty < w < \infty, \ 0 < k < \infty, \tag{3.18}$$

$$|Y^{(k)\prime}(w)| \leq k\sqrt{\varphi'(w)}Y^{(k)}(w), \quad -\infty < w < \infty, \ 0 < k < \infty \tag{3.19}$$

and

$$Y^{(k)}(w) \leq exp[k\int_{w_*}^{w} \sqrt{\varphi'(\xi)}d\xi], \quad -\infty < w < \infty, \ 0 < k < \infty. \tag{3.20}$$

We will estimate the solution $\{w(x,t), v(x,t)\}$ of (3.3) by monitoring the evolution of

$$\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,t), v(x,t))dx.$$

In view of the convexity of $\bar{\eta}^{(k)}(w,v)$, (3.9) yields

$$\begin{aligned}\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,t), v(x,t))dx &\leq \int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w_0(x), v_0(x))dx \\ &+ \int_0^t \int_{-\infty}^{\infty} \bar{\eta}_v^{(k)}(w(x,\tau), v(x,\tau))\mathcal{F}[v](x,\tau)dxd\tau, \quad 0 \leq t \leq T,\end{aligned} \tag{3.21}$$

where, by (3.16),

$$\bar{\eta}^{(k)}(w,v) = \eta^{(k)}(w,v) - Y^{(k)}(0) - Y^{(k)\prime}(0)w \geq 0, \tag{3.22}$$

and

$$\bar{\eta}_v^{(k)}(w,v) = k(tanh\ kv)\eta^{(k)}(w,v). \tag{3.23}$$

Using (3.23), (3.22), (3.19), the Cauchy-Schwarz inequality and (2.4), a lengthy calculation shows that the last integral in (3.21) can be estimated as follows:

$$\begin{aligned}&\Big|\int_0^t \int_{-\infty}^{\infty} \bar{\eta}_v^{(k)}(w,v)\mathcal{F}[v]dxd\tau\Big| \\ &= \Big|\int_0^t \int_{-\infty}^{\infty} k(tanh\ kv)(\bar{\eta}^{(k)}(w,v) + Y^{(k)}(0) \\ &\quad + Y^{(k)\prime}(0)w)\mathcal{F}[v]dxd\tau\Big| \\ &\leq Ck^2Y^{(k)}(0)\int_0^t \int_{-\infty}^{\infty} (|w| + |v|)|\mathcal{F}[v]|dxd\tau \\ &\quad + k\int_0^t \int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w,v)|\mathcal{F}[v]|dxd\tau \\ &\leq Ck^2Y^{(k)}(0)\Big[\int_0^t \int_{-\infty}^{\infty} (w^2 + v^2)dxd\tau + \int_{-\infty}^{\infty} v_0^2(x)dx\Big] \\ &\quad + Ck\int_0^t M(\tau)\Big[\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,\tau), v(x,\tau))dx\Big]d\tau,\end{aligned} \tag{3.24}$$

where

$$M(t) := \sup_{x\in\mathbf{R}}\{1 + |v(x,t)| + \int_0^t |v(x,\tau)|d\tau\}, \quad 0 \leq t \leq T. \tag{3.25}$$

Combining (3.21), (3.24), (3.15), and then using the Gronwall equality yields the estimate

$$\begin{aligned}\int_{-\infty}^{\infty} &\bar{\eta}^{(k)}(w(x,t), v(x,t))dx \\ &\leq \{\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w_0(x), v_0(x))dx \\ &\quad + Ck^2 Y^{(k)}(0)\} exp\{Ck\int_0^t M(\tau)d\tau\}, \quad 0 \leq t \leq T.\end{aligned} \tag{3.26}$$

Raising both sides of (3.26) to the $1/k$ power, letting $k \to \infty$ and then taking the logarithm of both sides of the resulting inequality (see [32] for justification of this procedure), we conclude

$$\sup_{x\in R}\{1 + |w(x,t)| + |v(x,t)|\} \leq C + C\int_0^t M(\tau)d\tau, \quad 0 \leq t \leq T. \tag{3.27}$$

Substituting (3.25) into (3.27) yields

$$S(t) \leq C + C(1+T)\int_0^t S(\tau)d\tau, \tag{3.28}$$

where $S(t) := \sup_{x\in\mathbf{R}}\{1 + |w(x,t)| + |v(x,t)|\}$. Finally, integrating (3.28) we conclude

$$\sup_{x\in\mathbf{R}}\{1 + |w(x,t)| + |v(x,t)|\} \leq M, \quad 0 \leq t \leq T, \tag{3.29}$$

where the constant M depends on T but not on ε. Therefore, the solution $\{w(x,t), v(x,t)\}$ of (3.3) is bounded on $\mathbf{R} \times [0,T]$, uniformly in $\varepsilon > 0$. This completes the sketch of the proof of the a-priori estimates in Theorem 3.1. ∎

Our final task in proving Theorem 2.1 is to use Theorem 3.1 to justify passage to the limit as $\varepsilon \downarrow 0^+$ in the Cauchy problem (3.1). For $\varepsilon > 0$, let $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}$ be the solution of (3.1) on $Q_T := \mathbf{R} \times [0,T]$, with regularity properties as in Theorem 3.1. By virtue of (3.29) the family of functions $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$. There exist functions $w(x,t), v(x,t)$ and $\bar{\varphi}(x,t)$ in $L^\infty(Q_T)$ such that, along a subsequence, $w^\varepsilon(x,t) \overset{*}{\rightharpoonup} w(x,t)$, $v^\varepsilon(x,t) \overset{*}{\rightharpoonup} v(x,t)$, and $\varphi(w^\varepsilon(x,t)) \overset{*}{\rightharpoonup} \bar{\varphi}(x,t)$ in L^∞-weak star as $\varepsilon \downarrow 0$. The objective is to show that $\{w(x,t), v(x,t)\}$ is a solution of (2.1) in the *sense of distributions for* $(x,t) \in Q_T$.

The major obstacle to overcome is that, in general, nonlinear functions are not continuous under weak star convergence, and it does not follow that $\bar{\varphi}(x,t) = \varphi(w(x,t))$. Under weak-star convergence, such composite weak limits have been characterized by Tartar [40,41,42] as expected values of a family of probability measures $\nu_{(x,t)}$, called the Young measures. The relevant issue here is whether the Young measure reduces to a Dirac mass. Using the theory of compensated compactness and a class of entropy-entropy flux pairs introduced by Lax [20], DiPerna [13] proved the following key result which will be employed in the sequel.

Proposition 3.2 (DiPerna). *Let $\{w^\varepsilon, v^\varepsilon\} : Q_T \to \mathbf{R}$ be a collection of functions such that*

$$\|w^\varepsilon\|_{L^\infty(Q_T)} + \|v^\varepsilon\|_{L^\infty(Q_T)} \leq C,$$

where C is a constant independent of ε. Suppose also that for any smooth entropy-entropy flux pair $\{\eta(w,v), q(w,v)\}$ of (E) with φ satisfying (2.5),

$$\partial_t \eta(w^\varepsilon(x,t), v^\varepsilon(x,t)) + \partial_x q(w^\varepsilon(x,t), v^\varepsilon(x,t))$$

lies in a compact set of $H^{-1}_{loc}(Q_T)$. Then there exists a subsequence $\{w^{\varepsilon'}, v^{\varepsilon'}\}$ and functions $w, v \in L^\infty(Q_T)$ such that

$$w^{\varepsilon'}(x,t) \to w(x,t),\ v^{\varepsilon'}(x,t) \to v(x,t),\ \text{a.e. } for\ (x,t) \in Q_T$$

as $\varepsilon' \downarrow 0$.

We apply Proposition 3.2 to the family $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}_{\varepsilon>0}$ of solutions of (3.1) (which also satisfy (3.3)). A straightforward calculation using (3.3) and (3.5) yields

$$\begin{aligned}
&\partial_t \eta(w^\varepsilon, v^\varepsilon) + \partial_x q(w^\varepsilon, v^\varepsilon) \\
&\quad = \varepsilon^{1/2}\partial_x(\varepsilon^{1/2}\eta_w(w^\varepsilon, v^\varepsilon)w^\varepsilon_x + \varepsilon^{1/2}\eta_v(w^\varepsilon, v^\varepsilon)v^\varepsilon_x) \\
&\quad\quad - \varepsilon[\eta_{ww}(w^\varepsilon, v^\varepsilon)(w^\varepsilon_x)^2 + 2\eta_{wv}(w^\varepsilon, v^\varepsilon)w^\varepsilon_x v^\varepsilon_x + \eta_{vv}(w^\varepsilon, v^\varepsilon)(v^\varepsilon_x)^2] \\
&\quad\quad + \eta_v(w^\varepsilon, v^\varepsilon)\mathcal{F}[v^\varepsilon] := I_1 - I_2 + I_3.
\end{aligned} \tag{3.30}$$

The a-priori estimates (3.15) and (3.29) imply that the family I_1 converges to 0 and is thereby compact in $H^{-1}(Q_T)$, while, the family I_2 is bounded in $L^1(Q_T)$. In addition,

$$\partial_t \eta(w^\varepsilon, v^\varepsilon) + \partial_x q(w^\varepsilon, v^\varepsilon) - I_3 \tag{3.31}$$

lies in a bounded set of $W^{-1,\infty}(Q_T)$. Using a lemma of Murat [28] we deduce that (3.31) lies in a compact set of $H^{-1}_{loc}(Q_T)$. Finally, since I_3 lies in a bounded set of $L^2(Q_T)$ and thus in a compact set of $H^{-1}_{loc}(Q_T)$, the left side of (3.30) also lies in a compact set of $H^{-1}_{loc}(Q_T)$. Thus, Proposition 3.2 implies that

$$w^{\varepsilon'}(x,t) \to w(x,t),\ v^{\varepsilon'}(x,t) \to v(x,t) \text{ a.e. in } Q_T,$$

along a subsequence $\varepsilon' \downarrow 0$, and permits passage to the limit $\varepsilon \downarrow 0$ in (3.1) in the sense of distributions. The pair of functions $\{w(x,t), v(x,t)\}$ belongs to $L^\infty([0,T]; L^2(\mathbf{R})) \cap L^\infty(Q_T)$ and is a weak solution of (2.1) in Q_T. This completes the sketch of the proof of Theorem 2.1. ∎

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